

Note on the Fusion Map and Hopf Algebras

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September 3, 2009

Abstract

We discuss an inversion property of the fusion map associated to many semibialgebras. Please note that a characterisation of Hopf k -algebras has been added at the end of this version.

Let $\mathcal{C} = (\mathcal{C}, \otimes, I, c)$ be a symmetric (or just braided) monoidal category. A Von Neumann “core” in \mathcal{C} is firstly a semibialgebra in \mathcal{C} , that is, an object A in \mathcal{C} with an associative multiplication:

$$\mu : A \otimes A \longrightarrow A$$

($\mu_3 = \mu(1 \otimes \mu) = \mu(\mu \otimes 1) : A \otimes A \otimes A \longrightarrow A$) and a coassociative comultiplication:

$$\delta : A \longrightarrow A \otimes A$$

($\delta_3 = (1 \otimes \delta)\delta = (\delta \otimes 1)\delta : A \longrightarrow A \otimes A \otimes A$) such that:

$$\delta\mu = (\mu \otimes \mu)(1 \otimes c \otimes 1)(\delta \otimes \delta) : A \otimes A \longrightarrow A \otimes A$$

It is also equipped with an endomorphism

$$S : A \longrightarrow A$$

in \mathcal{C} such that:

$$\mu_3(1 \otimes S \otimes 1)\delta_3 = 1 : A \longrightarrow A$$

The name “Von Neumann core” stems partly from the notion of a Von Neumann regular semigroup, which is then precisely a VN-core in **Set**, while the free vector space on it is a particular type of VN-core in **Vect**, and partly from the properties of the paths which generate a (row-finite) graph algebra[5].

The fusion map

$$f = (1 \otimes \mu)(\delta \otimes 1) : A \otimes A \longrightarrow A \otimes A$$

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then satisfies the fusion equation by the semibialgebra axiom of A (see [6]), and if we set:

$$g = (1 \otimes \mu)(1 \otimes S \otimes 1)(\delta \otimes 1)$$

as a tentative “inverse” to f , then we get the following (partial) results:

Proposition 1. [3] $fgf = f$ for any VN-core.

Proof. Define the (left) Fourier transform $l(\alpha)$ of a map $\alpha : A \longrightarrow B$ to be the composite

$$A \otimes B \xrightarrow{\delta \otimes 1} A \otimes A \otimes B \xrightarrow{1 \otimes \alpha \otimes 1} A \otimes B \otimes B \xrightarrow{1 \otimes \mu} A \otimes B$$

Then $l(\alpha \star \beta) = l(\alpha)l(\beta)$, where \star is the convolution $\alpha \star \beta = \mu(\alpha \otimes \beta)\delta$ of two maps α and β from A to B . Thus:

$$\begin{aligned} fgf &= l(1)l(S)l(1) \\ &= l(1 \star S \star 1) \\ &= l(1) \\ &= f \end{aligned}$$

since $1 \star S \star 1 = 1$ by the definition of VN-core. \square

Proposition 2. $fgf = g$ if $S^2 = 1$ and S is either an antipode or a coantipode.

The proof is straightforward.

Recall that a VN-core is called “unital” [1] if it satisfies the (stronger) axiom

$$1 \otimes \eta = (1 \otimes \mu)(1 \otimes S \otimes 1)\delta_3 : A \longrightarrow A \otimes A$$

where A is assumed to have the unit $\eta : I \longrightarrow A$. (A unital VN-core in $\mathcal{C} = \mathbf{Set}$ is precisely a group).

Proposition 3. [1] $gf = 1$ for any unital VN-core.

Note that, in general, if for a map f there exists a map g with $fgf = f$, then we can always find a map h with $fhf = f$ and $hfh = h$ provided idempotents split in \mathcal{C} .

A semibialgebra is called a very weak bialgebra in [1] if it also has both a unit $\eta : I \longrightarrow A$ ($\mu(1 \otimes \eta) = \mu(\eta \otimes 1) = 1$) and a counit $\epsilon : A \longrightarrow I$ ($(1 \otimes \epsilon)\delta = (\epsilon \otimes 1)\delta = 1$). A very weak bialgebra A is then called a very weak Hopf algebra if it is equipped with a map $S : A \longrightarrow A$ satisfying the axioms:

$$\begin{aligned} \mu(S \otimes 1)\delta &= t := (1 \otimes \epsilon\mu)(c \otimes 1)(1 \otimes \delta\eta) \\ \mu(1 \otimes S)\delta &= r := (\epsilon\mu \otimes 1)(1 \otimes c^{-1})(\delta\eta \otimes 1) \\ \mu_3(S \otimes 1 \otimes S)\delta_3 &= S \end{aligned}$$

Hence $S \star 1 \star S = S$ so that $fgf = g$ and, as a consequence of the semibialgebra axiom, we have $1 \star t = 1$ (see [4]) whence $1 \star S \star 1 = 1$ so that $fgf = f$ (using $S \star 1 = t$ by the first axiom).

Remark 1. There should be some form of reconstruction theorem for VN-cores, involving bimonoidal functors (U, r, r_0, i, i_0) for which $ri = 1$ (cf. [1]).

EXAMPLE: Suppose that $(A, \mu, \delta, \eta, \epsilon)$ is a bialgebra for which δ is *not* known to be coassociative, and suppose that A is also equipped with a map $S : A \longrightarrow A$ (not necessarily an antipode, say), and invertible elements $\alpha : I \longrightarrow A$ and $\beta : I \longrightarrow A$ such that the standard Drinfel'd axioms hold, namely:

$$\begin{aligned}\mu_3(S \otimes \alpha \otimes 1)\delta &= \alpha\epsilon \\ \mu_3(1 \otimes \beta \otimes S)\delta &= \beta\epsilon\end{aligned}$$

Proposition 4. This is a quasi-VN-core in the sense that both

$$\mu_3(1 \otimes S \otimes 1)(\delta \otimes 1)\delta = 1$$

and

$$\mu_3(1 \otimes S \otimes 1)(1 \otimes \delta)\delta = 1$$

Then at least

$$(1 \star S) \star 1 = 1 \star (S \star 1)$$

however here

$$l(\alpha \star 1) \neq l(\alpha)l(1)$$

and

$$l(1 \star \beta) \neq l(1)l(\beta)$$

in general.

Note that the two standard Drinfel'd conditions were still satisfied in the definition of a weak quasi-Hopf algebra (in the sense of Haring-Oldenburg et al. [2]).

Finally we note the following characterisation of Hopf k -algebras in terms of VN-cores, observing also that any VN-core A in \mathbf{Vect}_k can be completed to the VN-bialgebra $A \oplus k$.

Proposition 5. A VN-bialgebra in \mathbf{Vect}_k with S an antihomomorphism of k -algebras and $gfg = g$ is precisely a Hopf k -algebra.

The proof of this result uses the observation that the category of finite-dimensional representations of such a VN-bialgebra is left rigid with respect to the usual tensor product of left modules and duality. Thus Proposition 5 is related to the study of weak Hopf algebras, but only in that one can take the core of a given weak Hopf k -algebra A and complete it to a Hopf k -algebra structure $A \oplus k$.

Enquires (etc.) regarding this article can be made to the author through Micah McCurdy (Macquarie University), who kindly typed the manuscript.

References

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